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B.Sc. part - I(H), paper-I
Set Theory. Date- 16-04-2020.

Q.1. Define countable set and prove that countable union of countable sets is countable.

Soln: Countable Set: →
A set which is either finite or denumerable is said to be 'countable'.

Let any set X can be mapped one-one onto the set $\{1,2,\dots n\}$, where $n \in N$, then X is called a countable set, we also say that X has n elements.

In this case, if $X = \{1,2,\dots n\}$ is established by function f and $1 \le K \le n$, then under f, the image of $K \in \{1,2,\dots n\}$ is denoted as f_K.
Hence, we write $X = \{f_1, f_2, \dots f_n\}$

proof: Let $\{f_1, f_2, \dots f_n, \dots\}$ is a denumerable collection of sets where each f_n is denumerable.

Then, $f_n = \{a_{n1}, a_{n2}, a_{n3}, \dots, a_{nn}, \dots\}$ for each $n \in N$.

Thus, $f_1 = \{a_{11}, a_{12}, a_{13}, \dots, a_{1n}, \dots\}$
$f_2 = \{a_{21}, a_{22}, a_{23}, \dots, a_{2n}, \dots\}$
$f_3 = \{a_{31}, a_{32}, a_{33}, \dots a_{3n}, \dots\}$
$f_4 = \{a_{41}, a_{42}, a_{43}, \dots a_{4n}, \dots\}$
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Now, we can list all the elements of $f = \bigcup_{i=1}^{\infty} f_i$ in a sequence as follows.

Rest part of Q.1.

We first list a_{11}, and then we list a_{21}, a_{12} and then a_{31}, a_{22} and a_{13} and thus proceed diagonally. In this process we leave out those elements of any diagonal which already occur on any previous diagonal.

Thus, we can arrange the elements of $f = \bigcup_{i=1}^{\infty} f_i$ in a sequence by naming the objects as we meet them in the above procedure as $b_1, b_2, b_3, \cdot$ (Hence, $b_1 = a_{11}$). Thus $f = \bigcup_{i=1}^{\infty} f_i = \{b_1, b_2, b_3, \ldots\}$. Hence, f is denumerable and its cardinality is $\aleph_0$.

In this way countable union of countable sets is countable. proved

Q.2 Define a partial order relation on a set and illustrate the concept with two examples.

Soln Partial order relation: → Let X is any non-empty set and R is a relation in X then R is called a partial order relation in X when R is antisymmetric and transitive. If we write R as $\leq$, then

'$\leq$' is a partial order if

(I) For every $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$

II For every $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq$

The set $(X, \leq)$ is then called a partially ordered set.

Example I. For any set X, the inclusion relation $\subseteq$ is a partial order in $P(X)$.

Example II. In the set N of Natural Numbers the relation of divisibility '|' is a partial order, if $m | n$ we write $m \leq n$.

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B.Sc-part-II, paper-III
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Differential Calculus

Q.1 State and prove the Euler's Theorem on partial differentiation of homogeneous function of two independent variables.

Soln: Statement: → If $f(x,y)$ be a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof: Let $f(x,y) = Ax^{\alpha_1}y^{\beta_1} + Bx^{\alpha_2}y^{\beta_2} + Cx^{\alpha_3}y^{\beta_3} + \cdots$ —(I)

Where, $\alpha_1+\beta_1 = \alpha_2+\beta_2 = \alpha_3+\beta_3 = \cdots = n$ —— II

and $A, B, C, \cdots$ are constants i.e. independent of x and y.

Differentiating (I) partially with respect to x, we get

$$\frac{\partial f}{\partial x} = A\alpha_1 x^{\alpha_1-1}y^{\beta_1} + B\alpha_2 x^{\alpha_2-1}y^{\beta_2} + C\alpha_3 x^{\alpha_3-1}y^{\beta_3} + \cdots$$

$$\therefore x\frac{\partial f}{\partial x} = A\alpha_1 x^{\alpha_1}y^{\beta_1} + B\alpha_2 x^{\alpha_2}y^{\beta_2} + C\alpha_3 x^{\alpha_3}y^{\beta_3} + \cdots \quad \text{(III)}$$

Again, differentiating I partially w.r. to y, we get

$$\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1}y^{\beta_1-1} + B\beta_2 x^{\alpha_2}y^{\beta_2-1} + C\beta_3 x^{\alpha_3}y^{\beta_3-1} + \cdots$$

$$\therefore y\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1}y^{\beta_1} + B\beta_2 x^{\alpha_2}y^{\beta_2} + C\beta_3 x^{\alpha_3}y^{\beta_3} + \cdots \quad \text{(IV)}$$

Adding III and IV, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = Ax^{\alpha_1}y^{\beta_1}(\alpha_1+\beta_1) + Bx^{\alpha_2}y^{\beta_2}(\alpha_2+\beta_2) + Cx^{\alpha_3}y^{\beta_3}(\alpha_3+\beta_3)$$

$$= Ax^{\alpha_1}y^{\beta_1}n + Bx^{\alpha_2}y^{\beta_2}n + Cx^{\alpha_3}y^{\beta_3}n + \cdots$$

$$= n(Ax^{\alpha_1}y^{\beta_1} + Bx^{\alpha_2}y^{\beta_2} + Cx^{\alpha_3}y^{\beta_3} + \cdots) \quad \text{by II}$$

$$= nf(x,y) \quad [\text{from (I)}]$$

Proved

B.Sc - part - II(H), paper - III
Differential Calculus

Q.2. If $x^3+y^3-x^3y^2z=0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $x=y=1$.

Soln: We have, $x^3y^2z = x^3+y^3$

$$\therefore z = \frac{x^3+y^3}{x^3y^2} = \frac{1}{y^2}+\frac{y}{x^3}$$

$$\therefore \frac{\partial z}{\partial x} = -3\frac{y}{x^4}, \quad \frac{\partial z}{\partial y} = -\frac{2}{y^3}+\frac{1}{x^3}$$

$\therefore$ When $x=y=1$,

$$\frac{\partial z}{\partial x} = -3, \quad \frac{\partial z}{\partial y} = -2+1 = -1.$$

Q.3. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Soln. We have, $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$$\Rightarrow u = x^2(y-z)+y^2(z-x)+z^2(x-y) \quad \text{—— I}$$

Differentiating partially with respect to x, keeping y and z constants, we get

$$\frac{\partial u}{\partial x} = 2x(y-z)+y^2(-1)+z^2\cdot 1 = 2xy-2zx-y^2+z^2$$

Similarly, $\frac{\partial u}{\partial y} = x^2\cdot 1+2y(z-x)+z^2(-1) = x^2+2yz-2xy-z^2$

and $\frac{\partial u}{\partial z} = x^2(-1)+y^2\cdot 1+2z(x-y) = -x^2+y^2+2zx-2yz$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \cancel{2xy}-\cancel{2zx}-\cancel{y^2}+\cancel{z^2}+\cancel{x^2}+\cancel{2yz}-\cancel{2xy}-\cancel{z^2}-\cancel{x^2}+\cancel{y^2}+\cancel{2zx}-\cancel{2yz}$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 0$$

Proved.